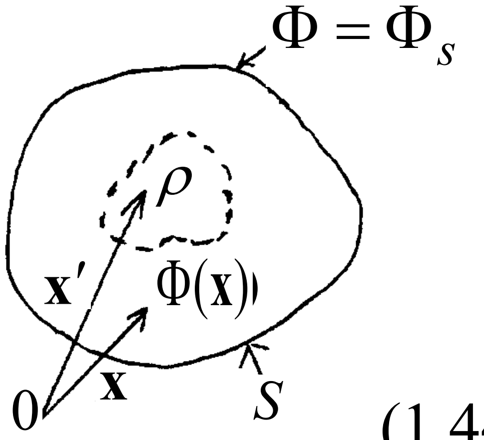
Chapter 2: Boundary-Value Problems in Electrostatics: I

Applications of Green's theorem

2.6 Green Function for the Sphere; General Solution for the Potential

The general electrostatic problem (upper figure):

$$\nabla^2\Phi(\mathbf{x}) = -\frac{1}{\varepsilon_0}\rho(\mathbf{x}) \text{ with b.c. } \Phi = \Phi_s$$



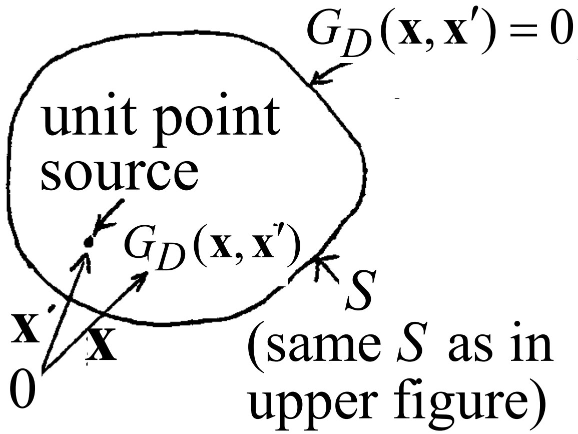
has the formal solution: (see Sec. 1.10)

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\mathbf{x}')G_D(\mathbf{x},\mathbf{x}')d^3x' - \frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\frac{\partial}{\partial n'}G_D(\mathbf{x},\mathbf{x}')da', \quad (1.44)$$

where the Green function $G_D(\mathbf{x},\mathbf{x}')$ is the solution of (lower figure)

$$\nabla^2 G_D(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}') \text{ with b.c. } G_D(\mathbf{x},\mathbf{x}') = 0 \text{ for } \mathbf{x} \text{ on } S$$

$G_D(\mathbf{x},\mathbf{x}')$ can be regarded as the potential due to a unit point source ($q \to 4\pi\varepsilon_0$, p. 64) at an arbitrary position $\mathbf{x}'$ inside the same surface S, but with the homogeneous b.c. $G_D(\mathbf{x},\mathbf{x}') = 0$ for $\mathbf{x}$ on S.



Example 1: Use (1.44) to find Φ due to a point charge q at $\mathbf{x}'=\mathbf{b}$ in infinite space.

$$\nabla^2\Phi(\mathbf{x}') = -\frac{q}{\varepsilon_0}\delta(\mathbf{x}'-\mathbf{b}) \text{ with b.c. } \Phi(\infty)=0$$

In order to use (1.44), we first obtain the Green function from

$$\nabla^2 G_D(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}') \text{ with } G_D(\mathbf{x},\mathbf{x}')=0 \text{ for } \mathbf{x} \text{ on S} \qquad (2)$$

The solution of (2) is $G_D(\mathbf{x},\mathbf{x}') = \dfrac{1}{|\mathbf{x}-\mathbf{x}'|}$

Sub. $q\delta(\mathbf{x}'-\mathbf{b})$ for $\rho(\mathbf{x}')$ and $1/|\mathbf{x}-\mathbf{x}'|$ for $G_D(\mathbf{x},\mathbf{x}')$ into (1.44)

$$\Rightarrow \Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \overbrace{\rho(\mathbf{x}')}^{q\delta(\mathbf{x}'-\mathbf{b})}\overbrace{G_D(\mathbf{x},\mathbf{x}')}^{\frac{1}{|\mathbf{x}-\mathbf{x}'|}} d^3x' - \frac{1}{4\pi}\oint_S \overbrace{\Phi(\mathbf{x}')}^{0}\frac{\partial G_D(\mathbf{x},\mathbf{x}')}{\partial n}da'$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{|\mathbf{x}-\mathbf{b}|}$$

Breaks a butterfly upon a wheel!

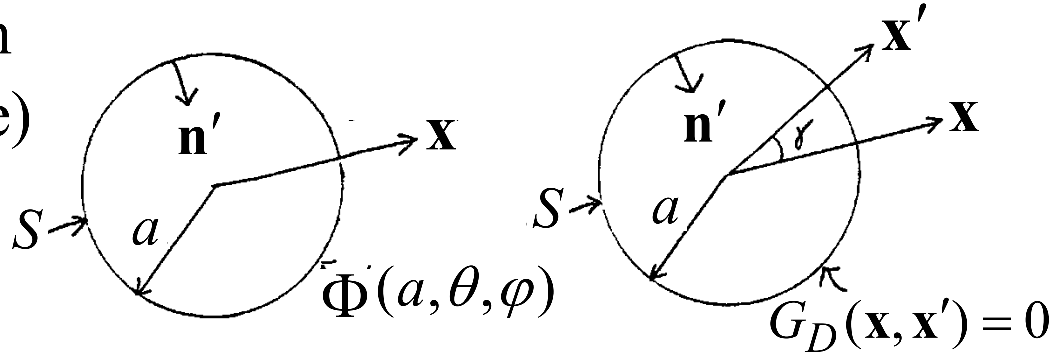
Example 2: $\nabla^2\Phi(\mathbf{x})=0$ with b.c. $\Phi(r=a)=\Phi(a,\theta,\varphi)$

Find $\Phi(\mathbf{x})$ in the region $r\geq a$ (see left figure).

First, find $G_D(\mathbf{x},\mathbf{x}')$ from the equation (see right figure)

$$\nabla^2 G_D(\mathbf{x},\mathbf{x}')=-4\pi\delta(\mathbf{x}-\mathbf{x}')$$

with $G_D(\mathbf{x},\mathbf{x}')=0$ on S.



Note: $\mathbf{n}$ points outward from the volume of interest.

This equation has the solution (see Sec. 2.2):

$$G_D(\mathbf{x},\mathbf{x}')=\frac{1}{|\mathbf{x}-\mathbf{x}'|}-\frac{a}{x'\left|\mathbf{x}-\frac{a^2}{x'^2}\mathbf{x}'\right|}$$

$\nabla^2 G_D(\mathbf{x},\mathbf{x}')=-4\pi\delta(\mathbf{x}-\mathbf{x}')$ in region of interest $(r\geq a)$

$$=\frac{1}{\left(x^2+x'^2-2xx'\cos\gamma\right)^{1/2}}-\frac{1}{\left(\frac{x^2x'^2}{a^2}+a^2-2xx'\cos\gamma\right)^{1/2}} \quad (3)$$

γ: angle between $\mathbf{x}$ and $\mathbf{x}'$

Note: $G_D(\mathbf{x},\mathbf{x}')=G_D(\mathbf{x}',\mathbf{x})$

$$\left.\frac{\partial G_D(\mathbf{x},\mathbf{x}')}{\partial n'}\right|_{x'=a} = -\left.\frac{\partial G_D(\mathbf{x},\mathbf{x}')}{\partial x'}\right|_{x'=a} = -\frac{(x^2-a^2)}{a(x^2+a^2-2ax\cos\gamma)^{3/2}} \quad (2.18)$$

Substituting (3) into (1.44)

$$\Rightarrow \Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \overbrace{\rho(\mathbf{x}')}^{=0} G_D(\mathbf{x},\mathbf{x}')d^3x' - \frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\overbrace{\frac{\partial G_D(\mathbf{x},\mathbf{x}')}{\partial n'}}da'$$

$$= \frac{1}{4\pi}\oint_S \Phi(a,\theta',\varphi')\frac{a(x^2-a^2)}{(x^2+a^2-2ax\cos\gamma)^{3/2}}d\Omega' \quad (2.19)$$

Questions:

1. In (3), we have $G_D(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{a}{x'\left|\mathbf{x}-a^2\mathbf{x}'/x'^2\right|}$ as a solution of $\nabla^2 G_D(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$. But $G_D(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|}$ apparently also satisfies the same equation. Does this violate the uniqueness thm.?
2. Can the solution of $\nabla^2 G_D(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}')$ be written in the form $G_D(\mathbf{x},\mathbf{x}') = G_D(\mathbf{x}-\mathbf{x}')$? why?

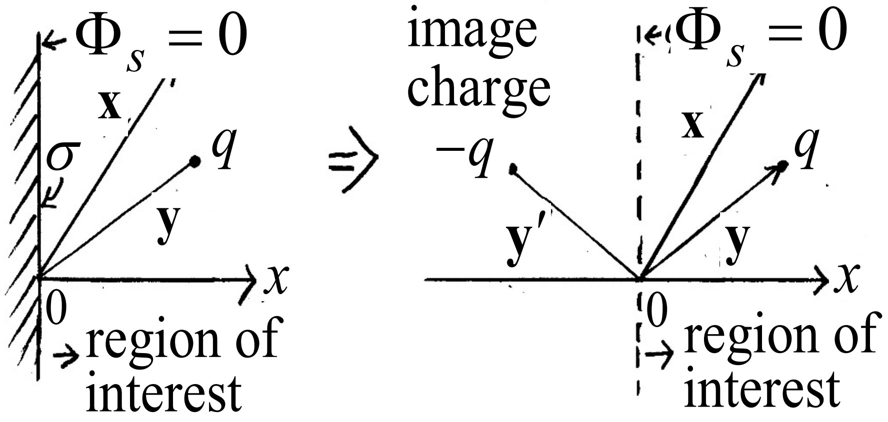
2.1 Method of Images

The method of images is not a general method. It works for the problems with simple geometries. Consider a point charge q located in front of an infinite and grounded plane conductor (see figure). The region of interest is $x \geq 0$ and Φ is governed by the Poisson equation:

$$\nabla^2 \Phi(\mathbf{x}) = -\frac{q}{\varepsilon_0}\delta(\mathbf{x}-\mathbf{y})$$

subject to the boundary condition

$$\Phi(x=0)=0.$$



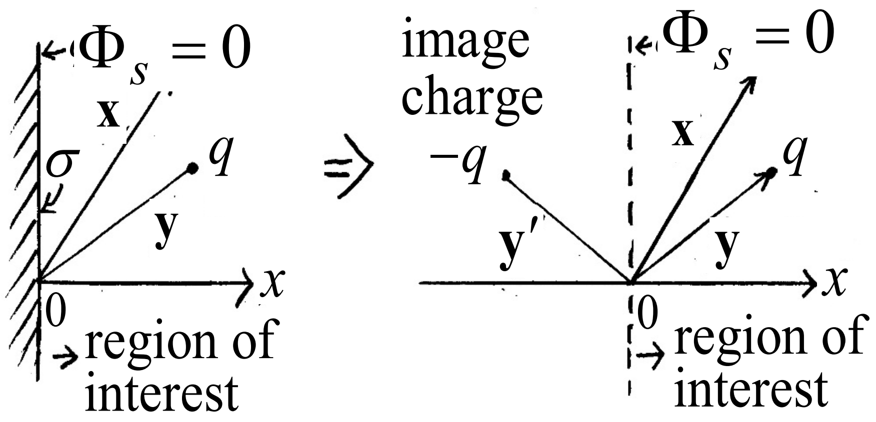
In order to maintain a zero potential on the conductor, surface charge will be induced (by q) on the conductor. We may simulate the effects of the surface charge with a hypothetical "image charge", $-q$, located symmetrically behind the conductor. Then,

2.1 Method of Images (*continued*)

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\mathbf{x}-\mathbf{y}|} - \frac{1}{|\mathbf{x}-\mathbf{y}'|}\right]$$

and, by symmetry, $\Phi(\mathbf{x})$ satisfies the boundary condition

$$\Phi(x=0)=0.$$



Operate $\Phi(\mathbf{x})$ with ∇^2

$$\Rightarrow \nabla^2\Phi(\mathbf{x}) = -\frac{q}{\varepsilon_0}[\delta(\mathbf{x}-\mathbf{y}) - \delta(\mathbf{x}-\mathbf{y}')] \quad (1)$$

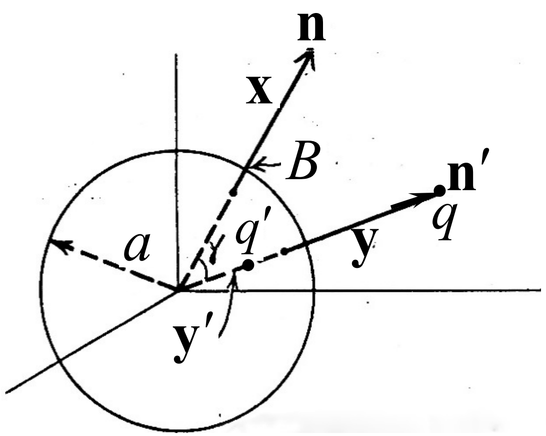
In the region of interest ($x \geq 0$), we have $\delta(\mathbf{x}-\mathbf{y}')=0$. Thus, $\Phi(\mathbf{x})$ obeys the original Poisson equation

$$\nabla^2\Phi(\mathbf{x}) = -\frac{q}{\varepsilon_0}\delta(\mathbf{x}-\mathbf{y})$$

[This shows that we must put the image charge outside the region of interest]

Since $\Phi(\mathbf{x})$ satisfies both the Poisson equation and the boundary condition in the region of interest, it is a solution. By the uniqueness theorem, it is the only solution. Note that the Poisson equation (1) and the solution $\Phi(\mathbf{x})$ are irrelevant outside the region of interest.

2.2 Point Charge in the Presence of a Grounded Conducting Sphere



Refer to the conducting sphere of radius a shown in the figure. Assume a point charge q is at $r = y$ $(> a)$. To find Φ for $r \geq a$, we put an image charge q' at $r = y'$ $(< a)$. Then,

$$\Phi(\mathbf{x}) = \frac{q/4\pi\varepsilon_0}{|\mathbf{x}-\mathbf{y}|} + \frac{q'/4\pi\varepsilon_0}{|\mathbf{x}-\mathbf{y}'|}$$

$$= \frac{q/4\pi\varepsilon_0}{|x\mathbf{n}-y\mathbf{n}'|} + \frac{q'/4\pi\varepsilon_0}{|x\mathbf{n}-y'\mathbf{n}'|}$$

Bounday condition requires

$$\Phi(a) = \frac{q/4\pi\varepsilon_0}{a\left|\mathbf{n}-\frac{y}{a}\mathbf{n}'\right|} + \frac{q'/4\pi\varepsilon_0}{y'\left|\frac{a}{y'}\mathbf{n}-\mathbf{n}'\right|} = 0$$

$$\Rightarrow \Phi(\mathbf{x}) = \frac{q/4\pi\varepsilon_0}{|\mathbf{x}-\mathbf{y}|} - \frac{aq/4\pi\varepsilon_0}{y\left|\mathbf{x}-\frac{a^2}{y^2}\mathbf{y}\right|}$$

First, let $\left|\mathbf{n}-\frac{y}{a}\mathbf{n}'\right| = \left|\frac{a}{y'}\mathbf{n}-\mathbf{n}'\right|$

$$1-2\frac{y}{a}\mathbf{n}\cdot\mathbf{n}'+(\tfrac{y}{a})^2 = (\tfrac{a}{y'})^2-2\frac{a}{y'}\mathbf{n}\cdot\mathbf{n}'+1$$

$$\Rightarrow \frac{y}{a} = \frac{a}{y'}, \text{ or } y' = \frac{a^2}{y},$$

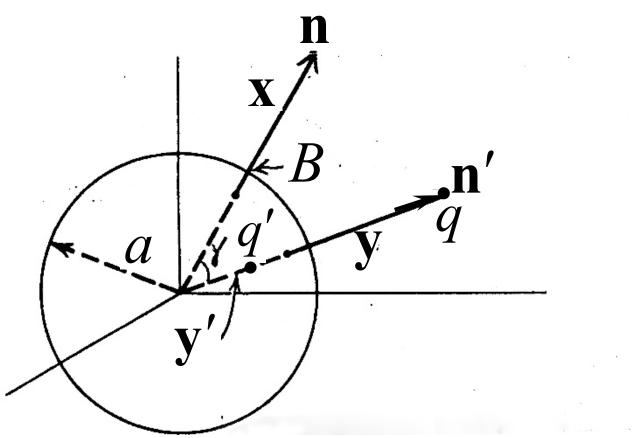
[Note: $y' < a$; hence, q' lies outside the region of interest.]

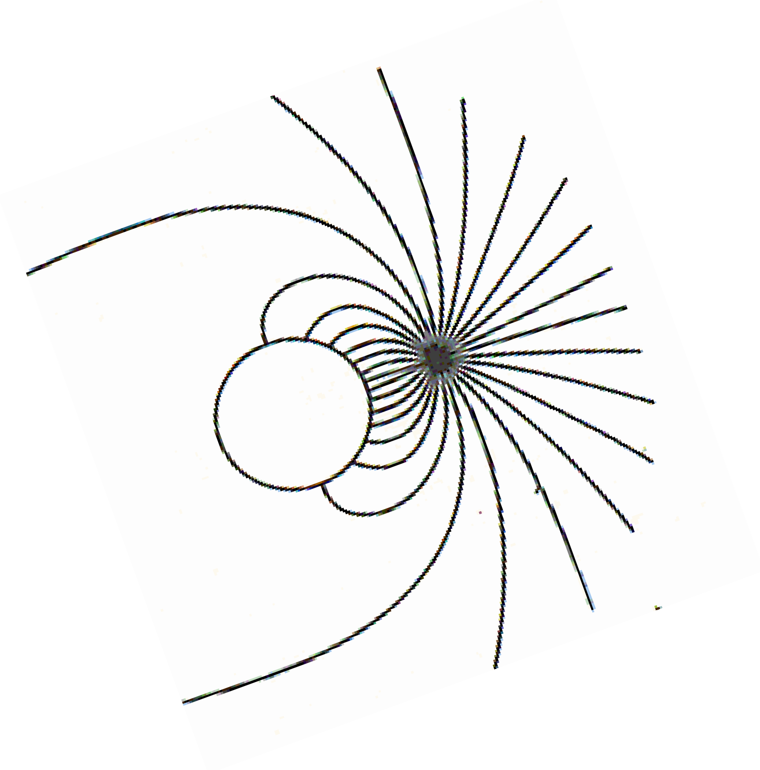
Next, set $\frac{q}{a} = -\frac{q'}{y'}$ so that RHS $= 0$.

This gives $q' = -\frac{y'}{a}q = -\frac{a}{y}q.$

Rewrite $\Phi(\mathbf{x}) = \dfrac{q/4\pi\varepsilon_0}{|\mathbf{x}-\mathbf{y}|} - \dfrac{aq/4\pi\varepsilon_0}{y\left|\mathbf{x}-\frac{a^2}{y^2}\mathbf{y}\right|}$ [This is equivalent to (2.1) and (2.4) of Jackson.]

In the region of interest ($r \geq a$), we have $\nabla^2\Phi(\mathbf{x}) = -\frac{q}{\varepsilon_0}\delta(\mathbf{x}-\mathbf{y})$. Thus, as in the case of the plane conductor, Φ satisfies the Poisson equation and the b.c. It is hence the only solution. The **E**-field lines are shown in the figure below.



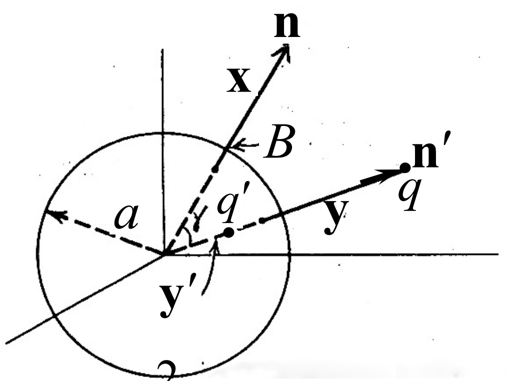


Surface charge density on the sphere: The solution for $\Phi(\mathbf{x})$ can be expressed in terms of scalars as

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{(x^2+y^2-2xy\cos\gamma)^{1/2}} - \frac{a}{y(x^2+\frac{a^4}{y^2}-2\frac{xa^2}{y}\cos\gamma)^{1/2}}\right]$$

where γ is the angle between $\mathbf{x}$ and $\mathbf{y}$.

By Gauss's law, the surface charge density at point B is



$$\sigma = \varepsilon_0 E_r(x=a) = -\varepsilon_0 \left.\frac{\partial\Phi}{\partial x}\right|_{x=a}$$

$$= \frac{q}{8\pi}\left[\frac{2a-2y\cos\gamma}{(a^2+y^2-2ay\cos\gamma)^{3/2}} - \frac{a(2a-2\frac{a^2}{y}\cos\gamma)}{y(a^2+\frac{a^4}{y^2}-2\frac{a^3}{y}\cos\gamma)^{3/2}}\right]$$

$$= \frac{-q}{4\pi a^2}\left(\frac{a}{y}\right)\frac{1-\frac{a^2}{y^2}}{(1+\frac{a^2}{y^2}-2\frac{a}{y}\cos\gamma)^{3/2}} \qquad (2.5)$$

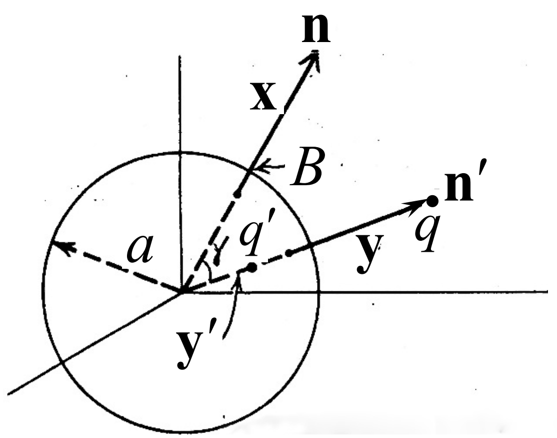
the law of cosines

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

$$(\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) = (x^2+y^2-2\mathbf{x}\cdot\mathbf{y})$$

Total charge on the sphere:

The total surface charge can be obtained by integrating σ over the spherical surface. However, it can be deduced from a simple argument: In the region $r \geq a$, the electric field due to the surface charge is exactly the same as that due to the image charge q'.



Hence, by Gauss's law, the total surface charge must be $q'(=-\frac{a}{y}q)$.

Force on q:

Since, at the position of charge q, the field produced by the image charge q' is the same as that produced by the surface charge, the force on q is the Coulomb force betwen q' and q.

$$\mathbf{F}=\frac{1}{4\pi\varepsilon_0}\frac{qq'}{(y-y')^2}\mathbf{n}'=\frac{-1}{4\pi\varepsilon_0}\frac{q(\frac{a}{y}q)}{(y-\frac{a^2}{y})^2}\mathbf{n}'=\frac{-1}{4\pi\varepsilon_0}\frac{q^2}{a^2}\frac{\left(\frac{a}{y}\right)^3}{(1-\frac{a^2}{y^2})^2}\mathbf{n}' \quad (2.6)$$

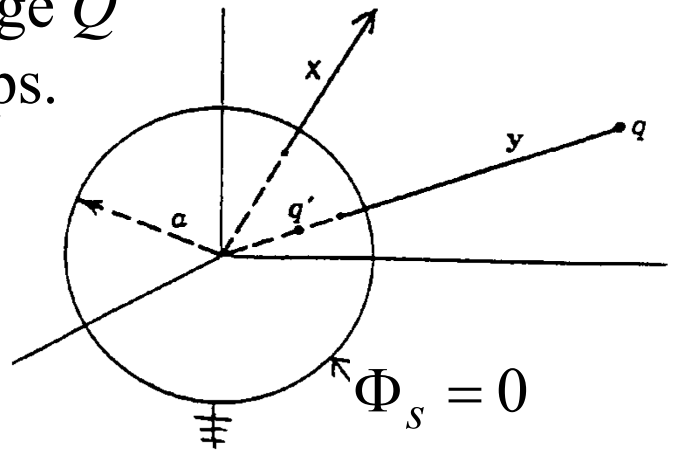
2.3 Point Charge in the Presence of a Charged, Insulated, Conducting Sphere (with Total Charge Q)

If the sphere is insulated with total charge Q on its surface, we may obtain Φ in two steps.

Step 1: Ground the sphere

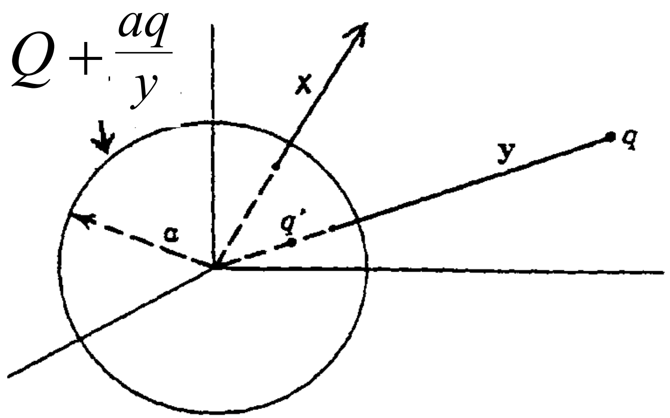
$\Rightarrow$ same problem as in Sec. 2.2

$$\Rightarrow \Phi(\mathbf{x}) = \frac{q/4\pi\varepsilon_0}{|\mathbf{x}-\mathbf{y}|} - \frac{aq/4\pi\varepsilon_0}{y\left|\mathbf{x}-a^2\mathbf{y}/y^2\right|}$$



with total surface charge $q' = -aq/y$.

Step 2: Disconnect the ground wire. Add $Q + aq/y$ to the sphere so that the total charge on the sphere is Q. Then, $Q + aq/y$ will be distributed uniformly on the surface because the charges are already in static equilibrium.



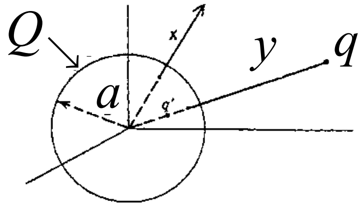
$$\Rightarrow \Phi \text{ due to } Q + aq/y \text{ is } \Phi(\mathbf{x}) = \frac{Q + aq/y}{4\pi\varepsilon_0 |\mathbf{x}|}$$

Hence, the total Φ is

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|\mathbf{x}-\mathbf{y}|} - \frac{aq}{y\left|\mathbf{x}-a^2\mathbf{y}/y^2\right|} + \frac{Q+aq/y}{|\mathbf{x}|}\right] \qquad (2.8)$$

The force on q is the force in (2.6) plus $\dfrac{q(Q+aq/y)}{4\pi\varepsilon_0}\dfrac{\mathbf{y}}{y^3}$ [force due to added charge]

$$\Rightarrow \mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{q}{y^2}\left[Q - \frac{qa^3(2y^2-a^2)}{y(y^2-a^2)^2}\right]\frac{\mathbf{y}}{y} \qquad (2.9)$$



$\Rightarrow$
- As $y \to \infty$, $F \to \dfrac{qQ}{4\pi\varepsilon_0 y^2}$ (Coulomb force between point charges)
- As $y \to a$, F is always attractive even if q and Q have the same sign.

Question: If there is an excess of electrons on the surface, why don't they leave the surface due to mutual repulsion?

(See p. 61 for a discussion on the work function of a metal.)

2.7 Conducting Spheres with Hemisphere…

(to be discussed in Sec. 3.3)

2.8 Orthogonal Functions and Expansions

Definition of Orthogonal Functions : xi [ksi]

Consider a set of real or complex functions $U_n(\xi)$ $(n = 1, 2, \cdots)$ which are square integrable on the interval $a \le \xi \le b$.

A function $f(x)$ is said to be square integrable if $\int_{-\infty}^{\infty} |f(x)|^2 dx$ is finite.

$U_n(\xi)$'s are

- orthogonal, if $\overbrace{\int_a^b U_n^*(\xi) U_m(\xi) d\xi}^{\text{inner product}} \begin{cases} = 0, & m \ne n \\ \ne 0, & m = n \end{cases}$
- orthonormal, if $\int_a^b U_n^*(\xi) U_m(\xi) d\xi = \delta_{mn} = \begin{cases} 0, & m \ne n \\ 1, & m = n \end{cases}$ **Kronecker delta**

KK[ˋænəlɔg]

Geometrical analogue: $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are an orthonormal set of unit vectors, i.e., $\mathbf{e}_m \cdot \mathbf{e}_n = \delta_{mn}$. By comparison, the dot product $\mathbf{e}_m \cdot \mathbf{e}_n$ is similar to the inner product . But the algebraic set $U_n(\xi)$ can be infinite in number.

Linearly Independent Functions :

The set of $U_n(\xi)$'s are said to be linearly independent if the only solution of $\sum_n a_n U_n(\xi) = 0$ (for every ξ in the range of $a \le \xi \le b$) is $a_n = 0$ for any n.

If a set of functions are orthogonal, they are also linearly independent.

Proof:

$$\sum_n a_n U_n(\xi) = 0$$

$$\Rightarrow \int_a^b \sum_n a_n U_n(\xi) U_m^*(\xi) d\xi = \sum_n a_n \overbrace{\int_a^b U_n(\xi) U_m^*(\xi) d\xi}^{=0,\text{ unless } n=m}$$

$$= a_m \int_a^b |U_m(\xi)|^2 \, d\xi = 0$$

$$\Rightarrow a_m = 0 \text{ for any } m.$$

Orthogonalization Procedure:

Orthogonality is a sufficient, but not necessary, condition for linear independence, i.e., linearly independent functions do not have to be orthogonal. However, they can be reconstructed into an orthogonal set by the Gram-Schmidt orthogonalization procedure.

Consider two vectors, $\mathbf{e}_x$ and $(\mathbf{e}_x + \mathbf{e}_y)$, as a simple example. These two vectors are not orthogonal, because $\mathbf{e}_x \cdot (\mathbf{e}_x + \mathbf{e}_y) \neq 0$, but are linearly independent because $a\mathbf{e}_x + b(\mathbf{e}_x + \mathbf{e}_y) = 0 \Rightarrow a = b = 0$.

We may form two new vectors as linear combinations of the old vectors, $\mathbf{e}_1 = \mathbf{e}_x$ and $\mathbf{e}_2 = \mathbf{e}_x + \mathbf{e}_y + \alpha\mathbf{e}_x$, and demand $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0$.

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = 0 \Rightarrow 1 + \alpha = 0 \Rightarrow \alpha = -1 \Rightarrow \mathbf{e}_2 = \mathbf{e}_y$$

The new set, $\mathbf{e}_1 (= \mathbf{e}_x)$ and $\mathbf{e}_2 (= \mathbf{e}_y)$, are thus orthogonal (as well as linearly independent).

The same procedure can be applied to algebraic functions.

Completeness of a Set of Functions :

Expand an arbitrary, square integrable function $f(\xi)$ in terms of a finite number (N) of functions in the orthonormal set $U_n(\xi)$,

$$f(\xi) \leftrightarrow \sum_{n=1}^{N} a_n U_n(\xi) \tag{2.30}$$

and define the mean square error (M_N) as

$$M_N \equiv \int_a^b \left| f(\xi) - \sum_{n=1}^{N} a_n U_n(\xi) \right|^2 d\xi.$$

If there exists a finite number N_0 such that for $N > N_0$ the mean square error M_N can be made smaller than any arbitrarily small positive quantity by proper choice of a_n's, then the set $U_n(\xi)$ is said to be complete and the series representation

$$\sum_{n=1}^{\infty} a_n U_n(\xi) = f(\xi) \tag{2.33}$$

is said to converge in the mean to $f(\xi)$.

Rewrite (2.33): $f(\xi)=\sum_{n=1}^{\infty} a_n U_n(\xi)$ (2.33)

Using the orthonormal property of $U_n(\xi)$'s, we get

$$a_n=\int_a^b U_n^*(\xi) f(\xi) d\xi \tag{2.32}$$

Change ξ in (2.32) to ξ' and substitute (2.32) into (2.33)

$$f(\xi)=\int_a^b\left[\sum_{n=1}^{\infty} U_n^*(\xi') U_n(\xi)\right] f(\xi') d\xi' \tag{2.34}$$

$f(\xi)$ is arbitrary $\Rightarrow \underbrace{\sum_{n=1}^{\infty} U_n^*(\xi') U_n(\xi)=\delta(\xi-\xi')}$ (2.35) **Dirac delta**

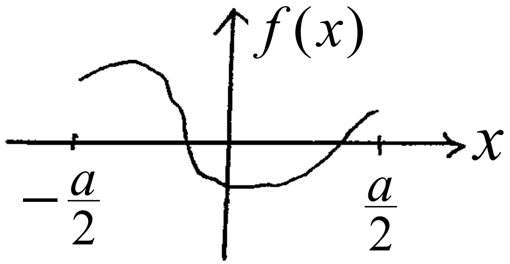
(completeness or closure relation)

Jackson, p. 68: "All orthonormal sets of functions normally occurring in mathematical physics have been proved to be complete." (This statement will be illustrated in Sec. 2.9.)

Fourier Series : example of complete set of orthogonal functions

Exponential representation of $f(x)$ *on the interval* $-\frac{a}{2} \leq x \leq \frac{a}{2}$:

$$\begin{cases} f(x) = \sum_{n=-\infty}^{\infty} a_n e^{ik_n x} \\ k_n = \frac{2\pi n}{a};\ a_n = \frac{1}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} f(x) e^{-ik_n x} dx \end{cases} \quad (4)$$



In (4), $f(x)$ is in general a complex function and, even when $f(x)$ is real, a_n is in general a complex constant.

In the case $f(x)$ is *real*, we have: $a_n = a_{-n}^*$

Proof: $f(x) = real \Rightarrow f(x) = f^*(x)$ $\quad$ $n \rightarrow -n$

$$\Rightarrow \sum_{n=-\infty}^{\infty} a_n e^{ik_n x} = \sum_{n=-\infty}^{\infty} a_n^* e^{-ik_n x} = \sum_{n=-\infty}^{\infty} a_{-n}^* e^{ik_n x}$$

$\Rightarrow a_n = a_{-n}^*$ (since $e^{ik_n x}$ is linearly independent)

Questions: 1. Why "$n = -\infty$ to ∞" instead of "$n = 0$ to ∞" ?

2. Why $k_n = 2\pi n/a$ instead of $k_n = \pi n/a$?

Sinusoidal representation of $f(x)$ *on the interval* $-\frac{a}{2} \le x \le \frac{a}{2}$:

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{ik_n x} = a_0 + \sum_{n=1}^{\infty} \left(a_n e^{ik_n x} + a_{-n} e^{-ik_n x} \right)$$

$$= a_0 + \sum_{n=1}^{\infty} \left[\left(a_n \cos k_n x + a_{-n} \cos k_n x \right) + i \left(a_n \sin k_n x - a_{-n} \sin k_n x \right) \right]$$

$$= a_0 + \sum_{n=1}^{\infty} \left(a_n + a_{-n} \right) \cos k_n x + \sum_{n=1}^{\infty} i \left(a_n - a_{-n} \right) \sin k_n x$$

$$\Rightarrow f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos k_n x + B_n \sin k_n x \right], \quad k_n = \frac{2\pi n}{a} \qquad (5)$$

where

Same as (2.36) and (2.37)

$$\begin{cases} A_n = a_n + a_{-n} = \frac{1}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} f(x) \underbrace{\left(e^{-ik_n x} + e^{ik_n x} \right)}_{2\cos k_n x} dx = \frac{2}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} f(x) \cos k_n x dx \quad (n = 0 \to \infty) \\ B_n = i \left(a_n - a_{-n} \right) = \frac{i}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} f(x) \underbrace{\left(e^{-ik_n x} - e^{ik_n x} \right)}_{-2i \sin k_n x} dx = \frac{2}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} f(x) \sin k_n x dx \quad (n = 1 \to \infty) \end{cases}$$

Discussion: It is often more convenient to express a physical quantity (a real number) in the exponential representation than in the sinusoidal representation, because the complex coefficient (a_n) of an exponential term carries twice the information of the real coefficient (A_n or B_n) of a sinusoidal term. For example, if

$$x(t) = \mathrm{Re}[ae^{i\omega t}]$$

is the displacement of a simple harmonic oscillator, the complex a contains both the magnitude and phase of the displacement. In the sinusoidal representation, the same quantity will be written

$$x(t) = A\cos(\omega t) + B\sin(\omega t).$$

Exponential terms are also easier to manipulate (such as multiplication and differentiation). This point will be further discussed in Ch. 7.

Fourier Transform :

If the interval becomes infinite ($a \to \infty$), we obtain the Fourier transform (see Jackson p.68).

$$\begin{cases} f(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} A(k) e^{ikx} dk & (2.44) \\ A(k) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x) e^{-ikx} dx & (2.45) \end{cases}$$

Change x to x' in (2.45) and substitute (2.45) into (2.44)

$$f(x) = \int_{-\infty}^{\infty} dx' f(x') \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk}_{\delta(x-x')}$$

$$\Rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk = \delta(x - x') \quad \text{[completeness relation]} \qquad (2.47)$$

Question 1: Does $A(k)$ contain any more or any less information than $f(x)$?

Rewrite (2.47): $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')}dk = \delta(x-x')$

Interchange x and k

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(k-k')x}dx = \delta(k-k'), \quad \text{[orthogonality condition]} \qquad (2.46)$$

Let $y = k - k'$ and substitute it into (2.46)

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixy}dx = \delta(y)$$

Since $\delta(y) = \delta(-y)$, we may write more generally,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\pm ixy}dx = \delta(y) \qquad (6)$$

Question 2 : Does a_n in $f(x) = \sum_n a_n e^{ik_n x}$ have the same dimension as $A(k)$ in $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)e^{ikx}dk$?

[assuming x is a dimensional quantity.]

There are two useful theorems concerning the Fourier integral:

(1) Parseval's theorem:

['parzefal]

The Parseval's theorem states $\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |A(k)|^2 dk \quad (7)$

Proof:

Rewrite the Fourier transform:
$$\begin{cases} f(x) = \dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{ikx} dk & (2.44) \\ A(k) = \dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx & (2.45) \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} f(x) f^*(x) dx$$

$$= \int_{-\infty}^{\infty} dx \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{ikx} dk \right] \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A^*(k') e^{-ik'x} dk' \right]$$

$$= \int_{-\infty}^{\infty} dk A(k) \int_{-\infty}^{\infty} dk' A^*(k') \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{i(k-k')x}}_{\delta(k-k')} = \int_{-\infty}^{\infty} |A(k)|^2 dk$$

(2) Convolution theorem: (Wikipedia: Linear time-invariant system)

The convolution theorem states

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}[\underbrace{\int_{-\infty}^{\infty} f_1(x-\xi)f_2(\xi)d\xi}]e^{-ikx}dx = A_1(k)A_2(k) \qquad (8)$$

This is called the convolution of $f_1(x)$ and $f_2(x)$

where the factor $1/2\pi$ follows the convention in (2.44) and (2.45).

Proof: LHS of (8) $= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_2(\xi)d\xi \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} f_1(x-\xi)e^{-ikx}dx}_{\text{Let } \eta=x-\xi \ (\Rightarrow dx=d\eta)}$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_2(\xi)d\xi \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_1(\eta)e^{-ik(\xi+\eta)}d\eta$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_2(\xi)e^{-ik\xi}d\xi \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_1(\eta)e^{-ik\eta}d\eta$$

$$= A_1(k)A_2(k)$$

2.9 Separation of Variables, Laplace Equation in Rectangular Coordinates

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \quad \begin{bmatrix}\text{Laplace equation in} \\ \text{Cartesian coordinates}\end{bmatrix} \qquad (2.48)$$

Let $\Phi(x, y, z) = X(x)Y(y)Z(z)$ (2.49)

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0 \qquad (2.50)$$

Since this equation holds for arbitrary values of x, y, and z, each of the three terms must be separately constant.

$$\Rightarrow \frac{d^2X}{dx^2} = -\alpha^2 X;\ \frac{d^2Y}{dy^2} = -\beta^2 Y;\ \frac{d^2Z}{dz^2} = \gamma^2 Z \text{ subject to } \gamma^2 = \alpha^2 + \beta^2$$

$$\Rightarrow X(x) = \begin{cases} e^{i\alpha x} \\ e^{-i\alpha x}\end{cases};\ Y(y) = \begin{cases} e^{i\beta y} \\ e^{-i\beta y}\end{cases};\ Z(z) = \begin{cases} e^{\gamma z} \\ e^{-\gamma z}\end{cases} \text{ with } \gamma = \sqrt{\alpha^2 + \beta^2}$$

Example: Find Φ inside a charge-free rectangular box (see figure) with the b.c. $\Phi\,(x,\ y,\ z=c)=V(x,\ y)$ and $\Phi=0$ on other sides.

$$X(x)=Ae^{i\alpha x}+Be^{-i\alpha x}$$

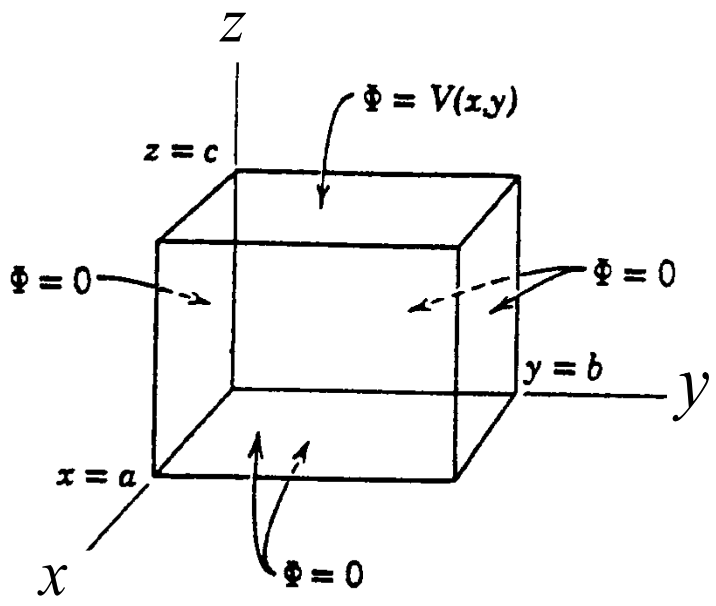
$$\begin{cases} X(0)=0 \Rightarrow B=-A \Rightarrow X=A(e^{i\alpha x}-e^{-i\alpha x})=A'\sin\alpha x \\ X(a)=0 \Rightarrow \alpha=\alpha_n=\dfrac{\pi n}{a},\quad n=1,2,\ldots \end{cases}$$

$$\Rightarrow X(x)=\sum_{n=1}^{\infty} A_n \sin\alpha_n x$$

Similarly, $Y(y)=Ae^{i\beta y}+Be^{-i\beta y}$.

$Y(0)=0$ and $Y(b)=0$ give

$$Y(y)=\sum_{m=1}^{\infty} A_m \sin\beta_m y,\quad \beta_m=\frac{\pi m}{b}$$



Solution for Z: $Z(z)=Ae^{\gamma z}+Be^{-\gamma z}$

$$Z(0)=0 \Rightarrow B=-A \Rightarrow Z(z)=A(e^{\gamma z}-e^{-\gamma z})=A''\sinh\gamma z$$

$$\Rightarrow \Phi=\sum_{n,m=1}^{\infty} A_{nm}\sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm} z),\ \gamma_{nm}=\sqrt{\alpha_n^2+\beta_m^2}\quad (2.56)$$

To find A_{nm}, we apply the b.c. on the $z=c$ plane:

$$\Phi(x,y,z=c)=V(x,y)$$

$$\Rightarrow V(x,y)=\sum_{n,m=1}^{\infty} A_{nm}\sin(\alpha_n x)\sin(\beta_m y)\sinh(\gamma_{nm}c) \tag{2.57}$$

$$\Rightarrow A_{nm}=\frac{4}{ab\sinh(\gamma_{nm}c)}\int_0^a dx\int_0^b dy V(x,y)\sin(\alpha_n x)\sin(\beta_m y) \tag{2.58}$$

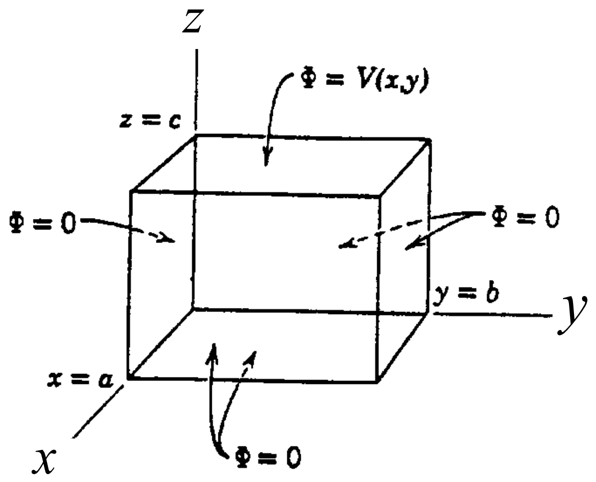
Questions:

1. The method of images is not a general method, but the method of expansion in orthogonal functions is. Why?
2. In electrostatics, only charges can produce Φ. In this problem, $\rho=0$, how can there be Φ ?
3. Can we find the surface charge distribution (σ) on the walls from the knowledge of Φ inside the box? If so, under what condition?

Discussion:

Rewrite (2.57): $V(x,y) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} c)$,

where $\alpha_n = \frac{\pi n}{a}$ and $\beta_m = \frac{\pi m}{b}$.



This is a good example to substantiate the following statement on physics ground: "All orthonormal sets of functions normally occurring in mathematical physics have been proved to be complete." (p. 68)

In (2.57), $\sin(\alpha_n x)$ and $\sin(\beta_m y)$ are orthogonal functions generated in a physics problem. Physically, we expect the problem to have a solution for any boundary condition on the surface $z = c$, i.e. for any function $V(x,y)$ specified in (2.57). Thus, $\sin(\alpha_n x)$ and and $\sin(\beta_m y)$ must each form a complete set in order to represent an arbitrary $V(x,y)$.

Homework of Chap. 2

Problem 2.1

A point charge q is brought to a position a distance d away from an infinite plane conductor held at zero potential. Use the method of images, find:

(a) the surface-charge density induced on the plane, and plot it;

(b) the force between the plane and the charge by using Coulomb's law for the force between the charge and its image;

(c) the total force acting on the plane by integrating $\sigma^2/2\varepsilon_0$ over the whole plane;

(d) the work necessary to remove the charge q from its position to infinity;

(e) the potential energy between the charge q and its image [compare the answer to part d and discuss].

(f) Find the answer to part d in electron volts for an electron originally one angstrom from the surface.

Problem 2.4

A point charge is placed a distance $d > R$ from the center of an equally charged, isolated, conducting sphere of radius R.

(a) Inside of what distance from the surface of the sphere is the point charge attracted rather than repelled by the charged sphere?

(b) What is the limiting value of the force of attraction when the point charge is located a distance? $a(=d-R)$ from the surface of the sphere, if $a \ll R$?

(c) What are the results for parts a and b if the charge on the sphere is twice (half) as large as the point charge, but still the same sign?

Homework of Chap. 2

Problem 2.5

(a)Show that the work done to remove the charge q from a distance $r > a$ to infinity against the force, Eq. (2.6), of a grounded conducting sphere is

$$W = \frac{q^2 a}{8\pi\varepsilon_0\left(r^2 - a^2\right)}$$

Relate this result to the electrostatic potential, Eq. (2.2), and the energy discussion of Section 1.11.

(b)Repeat the calculation of the work done to remove the charge q against the force, Eq. (2.9), of an isolated charged conducting sphere. Show that the work done is

$$W = \frac{1}{4\pi\varepsilon_0}\left[\frac{q^2 a}{2\left(r^2 - a^2\right)} - \frac{q^2 a}{2r^2} - \frac{qQ}{r}\right]$$

Relate the work to the electrostatic potential, Eq. (2.8), and the energy discussion of Section 1.11.

Problem 2.9

An insulated, spherical, conducting shell of radius a is in a uniform electric field E_0. If the sphere is cut into two hemispheres by a plane perpendicular to the field, find the force required to prevent the hemispheres form separating

(a) if the shell is uncharged;

(b) if the total charge on the shell is Q.

Homework of Chap. 2

Problem 2.23

A hollow cube has conducting walls defined by six planes $x=0$, $y=0$, $z=0$, and $x=a$, $y=a$, $z=a$. The walls $z=0$ and $z=a$ are held at a constant potential V. The other four sides are at zero potential.

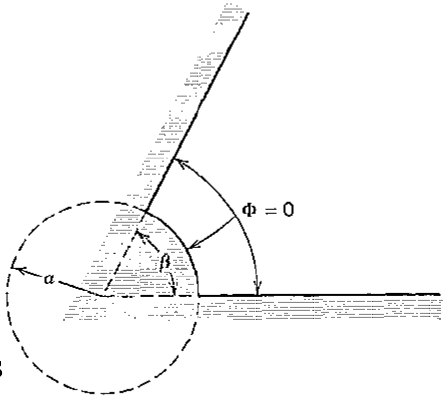
(a) Find the potential $\Phi(x, y, z)$ at any point inside the cube.

(b) Evaluate the potential at the center of the cube numerically, accurate to three significant figures. How many terms in the series is it necessary to keep in order to attain this accuracy? Compare your numerical result with the average value of the potential on the walls. See Problem 2.28.

(c) Find the surface-charge density on the surface $z=a$.

Problem 2.26

The two-dimensional region, $\rho \geq a$, $0 \leq \phi \leq \beta$, is bounded by conducting surfaces at $\phi=0$, $\rho=a$, and $\phi=\beta$ held at zero potential, as indicated in the sketch. At large ρ the potential is determined by some configuration of charges and/or conductors at fixed potentials.



(a) Write down a solution for the potential $\Phi(\rho, \phi)$ that satisfies the boundary conditions for finite ρ.

(b) Keeping only the lowest non-vanishing terms, calculate the electric field components E_ρ and E_ϕ and also the surface-charge densities $\sigma(\rho,0)$,$\sigma(\rho,\beta)$, and $\sigma(a,\phi)$ on the three boundary surfaces.

(c) Consider $\beta=\pi$ (a plane conductor with a half-cylinder of radius a on it). Show that far from the half-cylinder the lowest order terms of part b give a uniform electric field normal to the plane. Sketch the charge density on and in the neighborhood of the half-cylinder. For fixed electric field strength far from the plane, show that the total charge on the half-cylinder (actually charge per unit length in the z direction) is twice as large as would reside on a strip of width $2a$ in its absence. Show that the extra portion is drawn from regions of the plane nearby, so that the total charge on a strip of width large compared to a is the same whether the half-cylinder is there or not.

- Monday 15:30~17:30、物理館123、曾柏瑋
- Wednesday 15:30~17:30、物理館502A、劉禹賢
- Friday 15:30~17:30、物理館123、莊鎮銓

Homework of Chap. 2

Problem 2.3

A straight-line charge with constant linear charge density λ is located perpendicular to the $x-y$ plane in the first quadrant at $(x_0 - y_0)$. The intersecting planes $x=0, y \geq 0$ and $y=0, x \geq 0$ are conducting boundary surfaces held at zero potential. Consider the potential, fields, and surface charges in the first quadrant.

(a) The well-known potential for an isolated line charge at $(x_0 - y_0)$ is $\Phi(x, y) = (\lambda/4\pi\varepsilon_0)\,\text{In}(R^2/r^2)$, where $r^2 = (x-x_0)^2 + (y-y_0)^2$ and R is a constant. Determine the expression for the potential of the line charge in the presence of the intersecting planes. Verify explicitly that the potential and the tangential electric field vanish on the boundary surfaces.

(b) Determine the surface charge density σ on the plane $y=0, x \geq 0$. Plot σ/λ versus x for $(x_0=2, y_0=1), (x_0=1, y_0=1)$, and $(x_0=1, y_0=2)$.

(c) Show that the total charge density (pre unit length in z) on the plane $y=0, x \geq 0$, is.

$$Q_x = \frac{2}{\pi}\lambda \tan^{-1}\left(\frac{x_0}{y_0}\right)$$

What is the total charge on the plane?

(d) Show that far from the origin [$\rho \gg \rho_0$, where $\rho = \sqrt{(x^2+y^2)}$ and $\rho_0 = \sqrt{(x_0^2+y_0^2)}$ the leading term in the potential is

$$\Phi \to \Phi_{\text{asym}} = \frac{4\lambda}{\pi\varepsilon_0}\frac{(x_0 y_0)(xy)}{\rho^4}$$

Interpret.

Problem 2.2

Using the method of images, discuss the problem of a point charge q *inside* a hollow, grounded, conducting sphere of inner radius a, Find

(a) the potential inside the sphere;

(b) the induced surface-charge density;

(c) the magnitude and direction of the force acting on q.

(d) Is there any change in the solution if the sphere is kept at a fixed potential V? If the sphere has a total charge Q in its inner and outer surfaces?